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# Neglecting structural breaks when estimating and valuing dynamic correlations for asset allocation

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## Abstract

This paper assesses the econometric and economic value consequences of neglecting structural breaks in dynamic correlation models and in the context of asset allocation framework. It is shown that changes in the parameters of the conditional correlation process can lead to biased estimates of persistence. Monte Carlo simulations reveal that short-run persistence is downward biased while long-run persistence is severely upward biased, leading to spurious high persistence of shocks to conditional correlation. An application to stock returns supports these results and concludes that neglecting such structural shifts could lead to misleading decisions on portfolio diversification, hedging and risk management.

# 1 Introduction

Correlations among assets play a central role in both theoretical and empirical research, as understanding, estimating and interpreting comovements is crucial for market participants, institutions and policy makers. More specifically, for investors and managers, the optimal design of a well diversified portfolio and asset allocation decisions depend on a proper understanding of stock market correlations, while for policy makers correlations have implications on the stability of the economic and financial system. Furthermore, correlations determine hedge ratios in risk management and value-at-risk (VaR) measures of exposure to market comovements, required returns in asset pricing models, and asset allocations in market timing decisions. Therefore, a significant research in financial econometrics has focused on exploring tractable multivariate volatility models, in which the Dynamic Conditional Correlation (DCC) model (Engle, 2002) represents the benchmark specification for modelling volatilities and correlations of asset returns and for applications of optimal asset allocation over a large number of assets. Aielli (2006, 2013) argues that there are consistency problems with the DCC model and suggests the corrected DCC (cDCC) representation, although, in practice, both models perform similarly. Engle and Colacito (2006) and Colacito et al. (2011) take the asset allocation perspective to measure the economic value of modelling the conditional correlation structures. They show that the realized volatility is the smallest for the correctly specified covariance matrix for any vector of expected returns. Moreover, recent evidence by Buraschi *et al.* (2010) suggests that correlation risk might be more important than volatility risk in intertemporal risk problems.

Nevertheless, estimations of DCC models on financial returns and macroeconomic series indicate high persistence in the conditional correlation, i.e. the estimated persistence is close to one (see for example Bali and Engle, 2012, Brenner et al., 2009, Scruggs and Glabadanidis, 2013, Égert and Kočenda, 2010; Otranto, 2010; Savva et. al., 2009; Osborn et. al., 2008; Cappiello et. al., 2006, among others), where persistence is defined as the sum of the estimated parameters of the conditional covariance process. Similar behaviour has been observed in the univariate GARCH literature leading to the stylized fact that financial volatility exhibits high persistence (see for instance Engle and Bollerslev, 1986; Bollerslev and Engle, 1993; Baillie et al., 1996; Ding and Granger, 1996; Andersen and Bollerslev, 1997; Engle and Patton, 2001).

In addition, there has been evidence for the occurrence of structural breaks in financial time series as evidenced by Diebold (1986), Chu (1995), Bos et.al. (1999), Andreou and Ghysels (2002), among others, which led scholars to investigate whether structural changes may confound persistence estimation in volatility models (see Lamoureux and Lastrapes, 1990, Hamilton and Susmel, 1994, Gray, 1996 and Francq et. al., 2001). Theoretical analyses by Mikosch and Starica (2004), Hillebrand (2005) and Krämer and Azamo (2007) show that the estimated persistence close to unity in GARCH models might be spurious when there are structural breaks that are neglected. Recently, Amado and Teräsvirta (2014) allow individual variances to vary smoothly over time in order to account for deterministic changes before estimating a dynamic conditional correlation model.

Structural breaks in correlations play an important role in assessing bull and bear markets, identifying financial crisis, examining changes in financial market integration after, for example, the introduction of the euro, defining the relationship between macroeconomic vari-

ables and identifying if the contagion effect defined as a significant increase in correlations between asset returns in different markets is spurious. Despite the growing literature using the DCC model, there is only informal evidence that accounting for structural breaks in conditional correlations reduces the high persistence. A structural break in the conditional correlations is defined as a shift in the parameters of the conditional covariance process. Moreover, such empirical studies do not test and detect formally for the presence of structural breaks in the conditional correlations. Cappiello et. al. (2006) analyze the behavior of international equities and government bonds and given the high persistence in correlation upon inspection of the fit correlation and data, they concluded that a break should be included in order to capture a structural break in correlations. The break point was set to January 1999, when the euro currency was introduced. After the inclusion of a break, the persistence reduced substantially. Similarly, Savva (2009) and Savva et. al. (2009) found lower persistence in international stock market's conditional correlation once a break is included. In addition to the aforementioned studies, van Dijk et. al. (2011), using a sample of daily exchange rate returns, confirm that allowing for structural breaks in correlations decreases the persistence of conditional correlations. Hence, it is important to take the analysis one step further to examine from a theoretical viewpoint the consequences of ignoring structural breaks in conditional correlations. In addition, from an empirical viewpoint, it is important to employ formal tests for structural breaks in the correlation matrix in order to detect such shifts, rather than considering such structural breaks as being exogenous. This is because, decisions based on conditional models that do not account for structural changes in correlations can lead to misleading inferences and poor performance. Moreover, the quantitative asset allocation research ignores structural breaks in the dynamic correlation structure that could lead to sub-optimal asset allocation strategies. The presence of structural breaks should be exploitable in an asset allocation program as the optimal equity portfolio could be different in the presence of structural breaks compared to an optimal portfolio with stability in the dynamic correlation.

This paper examines the theoretical and empirical consequences of not allowing for structural breaks in the conditional correlation process on the persistence of the conditional correlation and on the economic value of modelling correlation structures. Firstly, it is established analytically that the estimated persistence of the conditional correlation process of the cDCC (or DCC) model converges to one when shifts in the unconditional correlation are not taken into account. Thus, high estimated persistence in conditional correlations is spurious when due to neglected parameter changes. Secondly, the Monte Carlo study confirms our theoretical results and reveals that the short-run persistence is downward biased while the long-run persistence is severely upward biased, leading to spurious high persistence of shocks to conditional correlation. This spurious estimated high persistence might have implications on the usefulness of the (c)DCC models for decisions about optimal portfolio diversification, hedging and risk management. Thirdly, an application to weekly returns on stock indices for United States, United Kingdom, Germany, France and Japan reveals that conditional correlations have undergone a significant structural break around 2000-2001 for the Euro Area market, and some of the countries' combinations, especially including United States, have undergone another or only a significant break around 2005-2006, which corresponds to the beginning of the collapse of the subprime market. Specifically, the procedures for testing and estimating structural breaks in the conditional covariance process proposed by

Aue et al (2009) are employed coupled with the methodology of detecting multiple change points in the correlation matrix of a sequence of random variables of Galeano and Wied (2015). Allowing for structural breaks in the conditional correlation, the persistence diminishes significantly, and thus estimating much lower values of half-life time of shocks to the conditional correlation. Furthermore, we examine the value of accounting for breaks in the conditional correlation structure in the context of asset allocation of Engle and Colacito (2006). The covariance estimators with or without structural breaks are compared using Diebold and Mariano tests indicating superior performance of the model that accounts for breaks.<sup>1</sup> Moreover, our empirical analysis shows that allowing for structural breaks improves investor's asset allocation strategy.

The rest of the paper is organized as follows. Section 2 discusses the DCC and cDCC models without and with the presence of structural breaks in the unconditional correlation matrix and establishes that not allowing for such structural breaks pushes the sum of the estimated persistence parameters of the conditional correlation process to unity. Section 3 presents Monte Carlo results to further analyse the finite sample properties of the (c)DCC estimators while section 4 offers some empirical evidence. Section 5 concludes.

## 2 Structural breaks in the DCC and cDCC models

Consider the following model

$$r_t = m_t(\theta) + \varepsilon_t, \quad t = 1, \dots, T \quad (1)$$

where  $r_t = (r_{1t}, \dots, r_{Kt})'$  is the vector of observed series,  $m_t$  is the conditional mean function, possibly non-linear and  $\theta$  is the true parameter vector. The error process  $\{\varepsilon_t, \mathcal{F}_t\}$  is a  $K \times 1$  vector of martingale difference sequences with conditional covariance matrix  $H_t$ , where  $\mathcal{F}_t$  is the information set that contains all past information up to and including time  $t$ . In the DCC model of Engle (2002), the conditional covariance  $H_t$  is factorized as

$$H_t = D_t R_t D_t \quad (2)$$

where  $D_t = \text{diag}(h_{1,t}^{1/2}, \dots, h_{K,t}^{1/2})$ ,  $h_{i,t}, i = 1, \dots, K$ , is the conditional variance of asset  $i$ , following a GARCH(1,1) process

$$h_{i,t} = \omega_i + \kappa_i \varepsilon_{i,t-1}^2 + \lambda_i h_{i,t-1}$$

with the usual restrictions for non-negativity and stationarity being imposed.

Engle (2002) introduces a transformation for the conditional covariance matrix in order to ensure its positive-definiteness and thus a resulting  $K \times K$  correlation matrix  $R_t$ , such as

$$R_t = \text{diag}(Q_t)^{-1/2} Q_t \text{diag}(Q_t)^{-1/2} \quad (3)$$

where  $Q_t$  has a GARCH type representation

$$Q_t = (1 - \alpha - \beta) \bar{Q} + \alpha z_{t-1} z'_{t-1} + \beta Q_{t-1} \quad (4)$$

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<sup>1</sup>Exceptions include the recent work of Kalotychou et al. (2014).

which defines the scalar DCC specification, where  $z_t = D_t^{-1}\varepsilon_t$  is the standardized disturbance vector,  $Q_t = \{q_{ij}\}$  and  $\bar{Q} = \{\rho_{ij}\}$  with  $\rho_{ii} = 1$  for all  $i$ . For  $R_t$  to be a proper correlation matrix, it suffices that  $Q_t$  is positive semi-definite with positive diagonal elements, thus  $\bar{Q}$  being positive semi-definite,  $\alpha > 0$ ,  $\beta \geq 0$  and  $\alpha + \beta < 1$  (see Ding and Engle, 2001). Many other specifications for the conditional correlation part of the DCC model have been proposed, see Tse and Tsui (2002), He and Terasvirta (2004), Hafner, Dijk and Franses (2006), Palandri (2009), Alexander (2002), Audrino and Barone-Adesi (2006) and Aielli (2006, 2013).

Aielli (2006, 2013) points out that in the DCC model  $\bar{Q} \neq E[z_t z_t']$ , and thus  $\bar{Q}$  ( $K \times K$ ) cannot be interpreted as the unconditional correlation or covariance matrix of  $z_t$ , which might lead to asymptotically biased estimators. He proposes the corrected DCC (cDCC) model by reformulating the correlation driving process as

$$Q_t = (1 - \alpha - \beta) \bar{Q} + \alpha z_{t-1}^* z_{t-1}^{*'} + \beta Q_{t-1} \quad (5)$$

where  $z_t^* = \text{diag}(Q_t)^{1/2} z_t$ , and thus  $E[z_t^* z_t^{*'} | \mathcal{F}_{t-1}] = Q_t$  and its unconditional properties follow from Lemma 1 presented in the Appendix.

The following analysis establishes the result that not accounting for structural changes in the unconditional correlation of  $z_t^*$  when estimating the cDCC (or DCC) model, the estimated persistence in the conditional covariance process of  $z_t^*$ , given by  $Q_t$ , converges to one. The arguments presented below for the multivariate GARCH models extend the ones proposed by Hillebrand (2005) for the univariate GARCH model and are based on the following assumption.

**Assumption 1**  $Q_t$  and  $z_t^*$  are observed without measurement error or at least it is independent of the estimated parameters in the (c)DCC model and vanishes with increasing sample size.

Incorporating  $k$  structural changes in the unconditional correlation, where  $k$  is a non-negative integer, the process  $Q_t$  in (5) within each segment can be replaced as in van Dijk et al (2011) by the following

$$Q_t = \bar{Q}_{t,l} + \alpha (z_{t-1}^* z_{t-1}^{*'} - \bar{Q}_{t-1,l}) + \beta (Q_{t-1,l} - \bar{Q}_{t-1,l}) \text{ for } t = T_{l-1} + 1, \dots, T_l \quad (6)$$

where

$$\bar{Q}_{t,l} = \bar{Q}_l 1(T_{l-1} < t \leq T_l)$$

and  $\bar{Q}_l = E(Q_t)$  for each segment  $l = 1, \dots, k+1$ , and let  $T_0 = 0$  and  $T_{k+1} = T$ .

Let  $E_{(l)}(Q_t) = E[Q_t | \mathcal{F}_{T_{l-1}}]$  denote the expectation of  $Q_t$  with respect to the information up to the initial value in segment  $l = 1, \dots, k+1$ . An application of Lemma 1 presented in the Appendix for each segment  $l = 1, \dots, k+1$  yields

$$\|E_{(l)}(Q_t) - \bar{Q}_l\| = o(1)_{T_l - T_{l-1}} \text{ for } T_{l-1} < t \leq T_l.$$

as  $T_l - T_{l-1} \rightarrow \infty$ . This result will be employed in proving the following theorem.

**Theorem 1** *If there are  $k$  structural changes in the data-generating process of the conditional covariance equation of the cDCC model given in (1)-(3) and (5), but these structural changes in the unconditional correlations are not allowed for in estimation, then under Assumption 1 and  $\alpha + \beta < 1$*

$$E_{(l)}(\hat{\alpha} + \hat{\beta}) = 1 \text{ as } T_l - T_{l-1} \rightarrow \infty.$$

The theorem establishes that neglected breaks in the conditional covariance process may induce spurious persistence of the conditional correlation process that could lead to misleading conclusion regarding inference and predictions. Proofs and other results are relegated to an Appendix.

### 3 Monte Carlo Analysis

This section presents Monte Carlo evidence on the finite sample properties of estimators in cDCC models and on the spurious persistence of the conditional correlation process when a structural break in the unconditional correlation is not allowed for in estimation. We also examine the consequences on the persistence of the correlation process when neglecting a structural break in the estimation of the volatility process.<sup>2</sup>

Firstly, we consider the finite sample properties of the estimators without a structural break. We firstly generate bivariate processes as given in (1)-(3) with  $m_t(\theta) = (0.1, 0.1)'$ , since estimation of the conditional mean specification does not influence our study. The GARCH processes are generated to be both highly persistent, both of low persistence and as well a combination of low and high persistence. In particular, we use  $\kappa_i \in \{0.1, 0.05, 0.09\}$ ,  $\lambda_i \in \{0.5, 0.9, 0.9\}$  and  $\omega_i = 1 - (\kappa_i + \lambda_i)$  for the specification of  $h_{i,t}$ ,  $i = 1, 2$ . All models are generated with the same persistence in the conditional correlation process. Specifically, the values of  $\alpha$  and  $\beta$  are set to 0.1 and 0.5, respectively, and  $\bar{Q}$  has element off diagonal  $\rho_{12} \in \{0.6, -0.4, 0.95\}$ .

Secondly, in order to examine the effects of not allowing for a structural break in the conditional correlation process, we contaminate the data with a deterministic shift in the unconditional correlation according to

$$\bar{Q}_t = \bar{Q}_1 1(t \leq [\tau T]) + \bar{Q}_2 1(t > [\tau T]), \quad (7)$$

where  $\tau$  is the break fraction, such that the conditional correlation process is generated as

$$Q_t = \bar{Q}_t + \alpha (z_{t-1}^* z_{t-1}^{*'} - \bar{Q}_{t-1}) + \beta (Q_{t-1} - \bar{Q}_{t-1})$$

for the cDCC model. The break fraction is set to  $\tau = 0.5$ ,  $\bar{Q}_1$  and  $\bar{Q}_2$  are matrices with elements off diagonal set to  $(\rho_{12,1}, \rho_{12,2}) = \{(0.4, 0.8, 0.05), (-0.4, 0.4, 0.95)\}$ . The values of all the other parameters are considered to be the same as for the case with no break. The results are also consistent to other break fractions. We also consider estimations for a higher dimension  $K$  using a more general specification allowing for the presence of a structural break

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<sup>2</sup>Monte Carlo simulations reveal similar results for the DCC model. The findings are available in the Supplementary Appendix.

in a subset or all the unconditional correlations. The spurious persistence is also examined when allowing for asymmetries in the conditional correlation process. In particular, we adopt the specification in quadratic form of Cappiello, Engle and Sheppard (2006) for the cDCC representation

$$Q_t = (\bar{Q}_t - A'\bar{Q}_tA - B'\bar{Q}_tB - \Gamma'\bar{G}_t\Gamma) + A'z_{t-1}^*z_{t-1}^{*'}A + B'Q_{t-1}B + \Gamma'\xi_{t-1}\xi_{t-1}'\Gamma$$

where  $\xi_t = 1(z_t^* < 0) \circ z_t^*$ ,  $\circ$  indicates the Hadamard product,  $A$ ,  $B$  and  $\Gamma$  are diagonal matrices with typical elements  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$ , which are set to  $\sqrt{0.1} = 0.316$ ,  $\sqrt{0.5} = 0.707$  and 0 or  $\sqrt{0.05} = 0.224$  when allowing for asymmetries, respectively, and  $\bar{G}_t$  is the sample covariance matrix of  $\xi_t$ .

Further, we investigate the finite sample properties of the cDCC estimators in the presence of a structural break in the volatility process that is not allowed for in estimation. For this we consider that  $h_{i,t}$  for  $i = 1, 2$  is generated according to

$$h_{i,t} = \omega_{i,1}1(t \leq [\tau T]) + \omega_{i,2}1(t > [\tau T]) + \kappa_i \varepsilon_{i,t-1}^2 + \lambda_i h_{i,t-1}$$

where the values of  $\omega_{i,1}$  and  $\omega_{i,2}$  in our Monte Carlo experiment are set to  $(\omega_{i,1}, \omega_{i,2}) = \{(0.4, 0.8), (0.05, 0.5)\}$ .

Each model is generated using  $T = 1100$  observations, from which the first 100 observations are discarded in order to avoid initialization problems. Each design is carried out with 1000 replications and estimation convergence is obtained in all cases.<sup>3</sup>

Table 1 reports the results when there is no break in the data generation processes for the unconditional correlation ( $\bar{Q}_t = \bar{Q}$ ). Specifically, we report the mean of the estimated parameters (Mean), the standard deviation of estimates from the true parameter values (Std. Dev.), the mean of estimated standard deviations (Est. Std. Dev.) and empirical rejection frequencies of the centred t-test (Centred t-test) that examine whether the estimates are significantly different from the true values at a 5% significance level.

As expected, the estimates are close to their true values. Moreover, the empirical rejection frequencies for the centred t-tests illustrate the consistency of the estimators. The empirical size of the centred t-test is slightly oversized for some of the parameters, including the ones in the conditional correlation process,  $\alpha$  and  $\beta$ . The values of the standard deviations of estimates from the true values are close to the ones of the mean of the estimated standard deviation, except for a few cases, where the estimate of the intercept in the GARCH models is upward biased.

Turning to the case where there is a structural break in the conditional correlation process that has not been taken into account when estimating bivariate cDCC models, these results are reported in Table 2. Although the parameters concerning the mean and volatility specifications are very close to their true values, this is not the case for parameters in the conditional covariance matrix. For all specifications, the estimator of  $\alpha$  reveals a downward bias, while the estimator of  $\beta$  is upward biased, with  $\hat{\alpha} + \hat{\beta} = 0.99$ , which indicates high persistence in the conditional correlation. Moreover, the standard deviations of estimates  $\hat{\alpha}$  and  $\hat{\beta}$  from their true values increase, in particular for the latter, while the means of the estimated standard

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<sup>3</sup>The parameters are estimated jointly within a quasi-maximum likelihood estimation framework.



deviations for these parameters decrease. The centred t-tests suggest that the estimates are significantly different from the true values in the presence of a break in the unconditional correlation. For the case where  $K = 4$ , similar results are obtained which are presented in Table 3.<sup>4</sup> Specifically, if there is a structural break in all the correlations, then all  $\alpha_i$  and  $\beta_i$ ,  $i = 1, \dots, K$  are affected with similar characteristics as for the case of  $K = 2$ . When the structural break is present in only one correlation process, then  $\alpha_i$  and  $\beta_i$  corresponding to the equations where structural breaks are present are affected as above; however, with the reduction in  $\alpha_i$  being less pronounced. Moreover, spurious high persistence in the conditional correlation also features when considering a more general specification allowing for asymmetries as presented in Table 4.

Moreover, we investigate the consequences of neglecting the presence of structural breaks in volatilities when estimating the bivariate cDCC models, with results reported in Table 5. We find that the estimates of the conditional correlations are not affected by possible breaks in volatilities. This result is in line with the findings of Amado and Teräsvirta (2014) who show that modeling the non-stationary component in variance has little effect on the DCC model. Finally, volatility parameters are in line with the related literature for univariate models (Lamoureux and Lastrapes, 1990; Hillebrand, 2005; Krämer and Azamo, 2007; among others) and the multivariate time varying correlation model of Amado and Teräsvirta (2014).

Therefore, the above analysis suggests that not allowing for a break in the unconditional correlation may lead to spurious persistence in the conditional correlation process<sup>5</sup>, although the presence of a break in the conditional volatility process does not contaminate the persistence in the conditional correlation.

## 4 Application on Structural Breaks and Optimal Asset Allocation

In this section we use real data to, firstly, examine whether structural breaks in correlation affect the parameter estimates by applying the test for detecting breaks in the conditional covariance process proposed by Aue et al (2009) coupled with the methodology of estimating multiple breaks of Galeano and Weid (2014). Secondly, we focus on the economic significance of the cDCC model allowing for breaks in the unconditional correlation in the context of asset allocation of Engle and Colacito (2006). Specifically, we apply the Diebold-Mariano (DM) test to examine whether there is an improvement in the performance of the cDCC with breaks. We use weekly returns on stock indices for the United States, the United Kingdom, Germany, France, and Japan over the period from 13 January 1996 to 14 May 2016 (1062 observations). As noted in various studies (see for instance Bartram et al., 2007; Capiello et al. 2006; Savva et al. 2009; among others) the correlation of the above series has undergone at least one significant structural break.<sup>6</sup>

<sup>4</sup>We have also run simulations for  $K = 10$  and the results are qualitatively the same. These findings have been relegated to the Supplementary Appendix.

<sup>5</sup>Similar results were obtained in the presence of two structural breaks in the unconditional correlation.

<sup>6</sup>The data is extracted from DataStream Global Equity Indices. The codes of those markets are TOTMKXX, where XX=US, UK, BD, FR and JP.

To obtain an accurate picture of the exact dates of breaks we employ the test of detecting breaks in the covariance matrix process proposed by Aue et al (2009). Specifically, Aue et al (2009) developed a nonparametric fluctuation test for a constant  $d$ -dimensional covariance matrix of the zero mean random vectors  $X_1, \dots, X_T$ , where  $X_k = (X_{k,1}, \dots, X_{k,d})'$ . The test is constructed based on a measure of fluctuations of the estimated covariance matrix

$$S_k = \frac{1}{\sqrt{T}} \left( \sum_{l=1}^k \text{vech}(X_l X_l') - \frac{k}{T} \sum_{l=1}^T \text{vech}(X_l X_l') \right)$$

for  $1 \leq k \leq T$ , where  $\text{vech}(\cdot)$  denotes the operator which stacks the columns on and below the diagonal of a  $d \times d$  matrix into a vector. The test statistic is then defined as  $\max_{1 \leq k \leq T} S_k' \hat{\Sigma}_T^{-1} S_k$ , where  $\hat{\Sigma}_T$  is a kernel based estimator. Aue et al (2009) established that under the null hypothesis of a constant covariance matrix, the limit null distribution of the test is  $\sup_{0 \leq s \leq 1} \sum_{j=1}^{d(d+1)/2} B_j^2(s)$ , where  $(B_j(s), s \in [0, 1]), j = 1, \dots, d(d+1)/2$  are independent Brownian bridge processes. If the null hypothesis is rejected, the unknown break point is estimated as  $\hat{T}_b = \arg \max_{1 \leq k \leq T} S_k' \hat{\Sigma}_T^{-1} S_k$ . Moreover, the procedure of Galeano and Weid (2014) is employed for detecting possible multiple breaks in the covariance matrix. Firstly, one tests for a change point in the whole sample. If a change point is detected, the sequence is split into two subsequences that are used to search for new change points. The procedure stops if no new change point is detected. Finally, once all break points are detected, a refinement step is employed to delete all possible false change points due to multiple testing and to estimate their location more accurately.

Table 6 presents the estimates of the cDCC models without and with structural breaks in the unconditional correlation. Specifically, column 1 refers to the combination of countries while columns 5-6 report the location of all possible structural breaks in the unconditional correlations that are estimated using the test of Aue et al (2009) coupled with the methodology of Galeano and Weid (2014) for detecting multiple change points in the conditional correlation process. For the bivariate combinations, the break dates for the Euro Area countries are around the period May 2000 - November 2001.<sup>7</sup> The break date around September 2005 - April 2006 for correlations between US and combinations of Germany, Japan and France corresponds to the beginning of the collapse of the subprime market. In the next step, we estimate the cDCC model with and without a break using the suggested location(s) of change as break points.

Columns 2 and 3 report the estimates of the cDCC model without and with accounting for structural breaks.<sup>8</sup> The estimation of the cDCC model without a break leads to highly persistent conditional correlations ( $\hat{\alpha} + \hat{\beta} = 0.990$  for almost all cases). More specifically, the half-life of shocks to the conditional correlations varies from 2.85 to 44.67 weeks (column 4).<sup>9</sup> In contrast, allowing for structural breaks in correlations decreases the persistence of conditional correlations, with the persistence varying between 0.454 to 0.978. This implies a substantial decline in the persistence of shocks to the conditional correlations, reducing the

<sup>7</sup>The test of Aue et al (2009) does not detect any break in the covariance matrix for the combination of US and France and it is excluded from Table 6.

<sup>8</sup>Similar results are obtained when the DCC specification is used. These results are available in the Supplementary Appendix.

<sup>9</sup>The half-life of shocks to the conditional correlations is given by  $HL = [\ln(0.5)/\ln(\hat{\alpha}^2 + \hat{\beta}^2)]$ .

half-life of shocks to a range from 0.43 to 16.73 weeks which is always below their counterparts of the cDCC without allowing for structural breaks.

The statistical significance of this change is tested by employing the likelihood ratio test (column 7) which suggests that in all cases this change is statistically significant. Therefore, the results indicate an improvement in fit due to allowing for breaks in correlations. Furthermore, the empirical application and tests highlight the importance of taking into account structural shifts in conditional correlation. Otherwise the results may lead to misleading inferences.

Figure 1, panels (a) and (b) report the estimated correlations between Germany and Japan and UK and France. These plots are representative for the majority of correlations and reveal potential benefits from employing the cDCC specification allowing for breaks over the original cDCC model. The former outperforms the latter mainly because it allows the long run correlation to change over time, reducing the overall persistence of correlation dynamics. The differences between the two specifications are more pronounced in Figure 2, where the difference between the estimated correlations is depicted.

To examine further the importance of structural break(s) in the cDCC process we apply a portfolio allocation strategy. It is well documented by Buraschi et al. (2010) that hedging demand tends to increase with the persistence of variance-covariance shocks. Hence, adopting a specification that does not properly model the variance-covariance matrix may lead to wrong asset allocation strategies.

The way to test the performance of variance-covariance obtained by cDCC specifications without and with breaks is in line with the methodology proposed by Engle and Colacito (2006) and applied by Colacito et al. (2011) for the case of MIDAS-DCC specification. While the readers are referred for further details to the aforementioned works, we briefly summarise the methodology below.

This approach assumes that an investor chooses optimal portfolio weights for  $N$  securities in order to minimise expected one day ahead portfolio variance subject to the constraint that portfolio weights must add up to some scalar,  $\bar{w}$

$$\min_{w_t} w_t' H_t w_t \quad \text{s.t.} \quad w_t' \mu_t = \mu_p \text{ and } w_t' \iota = 1 \quad (8)$$

where  $w_t$  is the vector of portfolio weights for time  $t$  chosen at time  $t - 1$ ,  $\mu_t$  is the assumed vector of excess returns with respect to the free-risk asset and  $\mu_p > 0$  is the required return and is set equal to 10%.  $H_t$  is the estimated one-period ahead conditional covariance matrix and  $\iota$  is an  $N \times 1$  vector of ones. Depending on the portfolio weights and the estimated  $H_t$ , the investor would end up with a certain amount of volatility. Assuming that the investor has two alternative conditional covariances (the one estimated by cDCC,  $H_{t\_cDCC}$  and the other by cDCC allowing for breaks  $H_{t\_breaks}$ ) the aim is to assess which portfolio is the most efficient.

Let portfolio returns attained according to each of the two estimators be denoted as

$$\pi_t^j = w_{t,j}' r_t, \quad \forall j \in \{cDCC, cDCC-breaks\} \quad (9)$$

where  $r_t$  stands for the demeaned vector of asset returns. Let the difference of the squared returns on the two portfolios be denoted as

$$u_t = (\pi_t^{cDCC})^2 - (\pi_t^{cDCC-breaks})^2. \quad (10)$$

By employing the Diebold and Mariano (1995) procedure (DM) we regress  $u_t$  on a constant and correct for heteroskedasticity. The null hypothesis of equal variance is simply a test that the mean of  $u_t$  is zero. Negative and significant values of the estimated parameters suggest that the cDCC specification allowing for breaks performs better than the alternative specification.

Portfolios of different sizes are constructed employing the covariance matrices estimated in Table 6, by selecting all possible combinations of two, three and four returns' series.<sup>10</sup> Column 8 in Table 6 reports the results of DM procedure. In almost all cases the cDCC specification that allows for breaks performs better than the standard cDCC suggesting that investors are better off taking possible breaks into account when they construct their portfolios. This result is in line with the findings of Colacito et al (2011) with our test being more significant for more combinations, suggesting that identifying and modeling breaks in conditional correlations leads to better performance compared to other specifications that may reduce persistence in correlations.

Finally, in columns 9 and 10 we assess the adequacy of the cDCC with breaks strategy on the basis of incremental utility relative to the simple cDCC strategy (see Fleming et al., 2001; and Kalotychou et al., 2014). The incremental value of correlation with breaks vis-a-vis the simple correlation is assessed by the return that would render an investor indifferent between the two strategies as follows

$$\sum_{t=1}^T \left[ (r_{cDCC\_breaks,t} - \Delta) - \frac{\gamma}{2(1+\gamma)} (r_{cDCC\_breaks,t} - \Delta)^2 \right] = \sum_{t=1}^T \left[ r_{cDCC,t} - \frac{\gamma}{2(1+\gamma)} r_{cDCC,t}^2 \right] \quad (11)$$

where  $r_{cDCC,t}$  and  $r_{cDCC\_breaks,t}$  represent the returns for the cDCC without and with breaks, respectively,  $\Delta$  denotes the weekly cost for the investor to adopt the strategy with breaks while  $\gamma$  denotes the degree of constant relative risk-aversion. Using the above equation we are able to calculate the performance fee (PF), for a given  $\gamma$ , as the average annualized fee (in basis points) an investor with quadratic utility is willing to pay to switch from the simple cDCC to a cDCC with breaks strategy. In all cases we find large and positive performance fees across all portfolios, suggesting that strategies that account for breaks in correlations have real economic value.

In addition to the above, we also employ the portfolio optimization strategy based on maximizing the expected portfolio return subject to a target conditional volatility:

$$\max_{w_t} w_t \mu_t \text{ s.t. } \sigma_p^2 = w_t' H_t w_t \text{ and } w_t' \iota = 1$$

where  $\sigma_p$  is the target level of risk. The results available in columns 11-13 of Table 6 suggest that the dynamic specification that accounts for breaks is able to generate higher reward-to-risk ratios than the specification without breaks. In all cases cDCC with breaks performs better according to the DM test than the alternative.

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<sup>10</sup>The choice of countries is identical to the example in Colacito et al. (2011).

## 5 Conclusion

We show analytically that the estimated persistence of the correlation process in (c)DCC models tends to converge to one when not allowing for the presence of structural breaks in the unconditional correlation. Our results extend the ones already established of spurious persistence in autoregression and conditional volatility models. Our Monte Carlo study confirms the spurious persistence in the conditional correlation process. Moreover, this spurious behaviour carries over to an empirical analysis to weekly returns on some major stock indices. It also reveals that the conditional correlation has undergone significant structural shifts which are determined by employing the tests for detecting change points in the covariance matrix of a sequence of random variables proposed by Aue et al (2009) coupled with the methodology of detecting multiple change points of Galeano and Wied (2014). After allowing for structural breaks in the unconditional correlation of the markets examined, the measure of the persistence in the conditional correlation diminishes significantly. Moreover, portfolio analysis suggests that accounting for breaks in conditional correlations is economically relevant. By applying the DM test, we find that the (c)DCC allowing for breaks outperform the (c)DCC specification that does not account for breaks.

Generally, our findings highlight the importance of taking possible structural breaks into account, when the conditional correlations are estimated. One possibility to alleviate the problem of structural breaks is to use regime switching models (such as the smooth transition model of Silvennoinen and Teräsvirta, 2009; or Markov switching specification of Baele, 2005, Pelletier, 2006, and Billio and Caporin, 2005). Although these models account for different behaviour in conditional correlations, in contrast to (c)DCC specification, they fail to assess the exact correlation at each point of time (since they assume constant correlations within the regime but different across regimes).

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## Appendix

**Lemma 1** *Let  $E_0(Q_t)$  denote the expected value of the stationary conditional covariance process  $Q_t$  in (4) given a start value  $Q_0$ . If  $\alpha + \beta < 1$  then  $\|E_0(Q_t) - E(Q_t)\| = o(1)_T$  holds for  $t = 1, \dots, T$  where  $E(Q_t) = \bar{Q}$  and  $o(1)_T \rightarrow 0$  as  $T \rightarrow \infty$ .*

**Proof of Lemma 1.** Firstly, the cDCC process can be written as

$$Q_t = \sum_{i=1}^{\infty} \beta^{i-1} [(1 - \alpha - \beta) \bar{Q} + \alpha z_{t-i}^* z_{t-i}^{*'}]$$

and

$$E_{t-1}(Q_t) = \sum_{i=1}^{\infty} \beta^{i-1} [(1 - \alpha - \beta) \bar{Q} + \alpha z_{t-i}^* z_{t-i}^{*'}].$$

Further

$$\begin{aligned}
E_{t-2}(Q_t) &= E_{t-2}(E_{t-1}(Q_t)) \\
&= \sum_{i=1}^{\infty} \beta^{i-1} [(1 - \alpha - \beta) \bar{Q} + \alpha E_{t-2}(z_{t-i}^* z_{t-i}^{*'})] \\
&= (1 - \alpha - \beta) \bar{Q} + \alpha E_{t-2}(z_{t-1}^* z_{t-1}^{*'}) + \sum_{i=2}^{\infty} \beta^{i-1} [(1 - \alpha - \beta) \bar{Q} + \alpha E_{t-2}(z_{t-i}^* z_{t-i}^{*'})] \\
&= (1 - \alpha - \beta) \bar{Q} + \alpha \sum_{i=1}^{\infty} \beta^{i-1} [(1 - \alpha - \beta) \bar{Q} + \alpha z_{t-i-1}^* z_{t-i-1}^{*'}] \\
&\quad + \sum_{i=2}^{\infty} \beta^{i-1} [(1 - \alpha - \beta) \bar{Q} + \alpha z_{t-i}^* z_{t-i}^{*'}] \\
&= (1 - \alpha - \beta) \bar{Q} + \alpha \sum_{i=1}^{\infty} \beta^{i-1} [(1 - \alpha - \beta) \bar{Q} + \alpha z_{t-i-1}^* z_{t-i-1}^{*'}] \\
&\quad + \beta \sum_{i=1}^{\infty} \beta^{i-1} [(1 - \alpha - \beta) \bar{Q} + \alpha z_{t-i-1}^* z_{t-i-1}^{*'}] \\
&= (1 - \alpha - \beta) \bar{Q} + (\alpha + \beta) \sum_{i=1}^{\infty} \beta^{i-1} [(1 - \alpha - \beta) \bar{Q} + \alpha z_{t-i-1}^* z_{t-i-1}^{*'}]
\end{aligned}$$

and by recursive substitution

$$E_{t-s}(Q_t) = (1 + (\alpha + \beta) + \dots + (\alpha + \beta)^{s-2}) (1 - \alpha - \beta) \bar{Q} \\ + (\alpha + \beta)^{s-1} \sum_{i=1}^{\infty} \beta^{i-1} [(1 - \alpha - \beta) \bar{Q} + \alpha z_{t-i-s+1}^* z_{t-i-s+1}^{*'}]$$

If  $\alpha + \beta < 1$ , then as  $s \rightarrow \infty$ ,  $E_{t-s}(Q_t) \rightarrow \bar{Q}$ . It also follows that conditional on a fixed initial value  $Q_0$  for the conditional correlation process such that  $s = t$ , then  $\|E_0(Q_t) - \bar{Q}\| = o(1)_T$ . ■

**Lemma 2** Let  $\tilde{Q} = T^{-1} \sum_{t=1}^T z_t^* z_t^{*'}$  denote the sample mean of the conditional covariance over the entire sample. In the presence of structural breaks and under Assumption 1,

$$\tilde{Q} = \frac{1}{T} \sum_{l=1}^{k+1} (T_l - T_{l-1}) \bar{Q}_{t,l} + o(1)_T \quad (12)$$

where  $\bar{Q}_{t,l}$  is the unconditional expected value of  $Q_t$  within segment  $l = 1, \dots, k+1$  and it is assumed that  $T_l - T_{l-1} \rightarrow \infty$ .

**Proof of Lemma 2.** The proof follows similar arguments as in Lemma 2 of Hillebrand (2005), by writing for each segment

$$z_t^* z_t^{*' } = E_{(l)}(Q_t) + R_t \quad (13)$$

where

$$\frac{1}{T_l - T_{l-1}} \sum_{t=T_{l-1}+1}^{T_l} \|R_t\| = o(1)_{T_l - T_{l-1}}$$

such that  $T_l - T_{l-1} \rightarrow \infty$  and applying Lemma 1. ■

**Proof of Theorem 1.** The  $(i, j)$ th element of the estimated conditional covariance matrix  $Q_t$  when neglecting the presence of structural breaks in the unconditional covariance, can be written as

$$q_{ij,t} - \widetilde{\rho}_{ij} = \hat{\alpha} (z_{i,t-1}^* z_{j,t-1}^* - \widetilde{\rho}_{ij}) + \hat{\beta} (q_{ij,t-1} - \widetilde{\rho}_{ij})$$

under Assumption 1, where hats denote evaluation at the estimators whereas  $\widetilde{\rho}_{ij}$  is the  $(i, j)$ th element of the sample covariance matrix  $T^{-1} \sum_{t=1}^T z_t z_t'$ . Taking expectations throughout with respect to the initial values within each segment  $l = 1, \dots, k+1$  yields

$$E_{(l)}(q_{ij,t}) - E_{(l)}(\widetilde{\rho}_{ij}) = E_{(l)}(\hat{\alpha} z_{i,t-1}^* z_{j,t-1}^*) - E_{(l)}(\hat{\alpha} \widetilde{\rho}_{ij}) + E_{(l)}(\hat{\beta} q_{ij,t-1}) - E_{(l)}(\hat{\beta} \widetilde{\rho}_{ij}) \quad (14)$$

Applying Lemma 1, substituting  $\widetilde{\rho}_{ij}$  from (12) and using the fact that the influence of a single realization of  $z_{i,t-1}^* z_{j,t-1}^*$  and  $q_{ij,t}$  on the estimated parameters  $\hat{\alpha}$  and  $\hat{\beta}$  vanishes as  $T \rightarrow \infty$ , (14) becomes

$$\rho_{(ij)t,l} - \frac{1}{T} \sum_{m=1}^{k+1} (T_m - T_{m-1}) \rho_{(ij)t,m} + o(1)_T = E_{(l)}(\hat{\alpha}) E_{(l)}(z_{i,t-1}^* z_{j,t-1}^*) - E_{(l)}(\hat{\alpha} \widetilde{\rho}_{ij}) \\ + E_{(l)}(\hat{\beta}) E_{(l)}(q_{ij,t-1}) - E_{(l)}(\hat{\beta} \widetilde{\rho}_{ij}) + o(1)_T$$

where  $\rho_{(ij)t,l}$  represents the  $(i, j)$ th element of  $\bar{Q}_{t,l}$ . Further, employing Lemma 1 and 2

$$\begin{aligned}
& \rho_{(ij)t,l} - \frac{1}{T} \sum_{m=1}^{k+1} (T_m - T_{m-1}) \rho_{(ij)t,m} + o(1)_T \\
&= E_{(l)}(\hat{\alpha}) \left( \rho_{(ij)t-1,l} + (\alpha + \beta)^{t-T_{l-1}-2} c_{t-1,l} \right) \\
&- E_{(l)} \left[ \hat{\alpha} \left( \frac{1}{T} \sum_{m=1}^{k+1} (T_m - T_{m-1}) \rho_{(ij)t-1,m} + \frac{1}{T} \sum_{l=1}^{k+1} \sum_{t=T_{l-1}+1}^{T_l} (\alpha + \beta)^{t-T_{l-1}-2} c_{t-1,l} \right. \right. \\
&\left. \left. + \frac{1}{T} \sum_{l=1}^{k+1} \sum_{t=T_{l-1}+1}^{T_l} r_{t-1} \right) \right] + E_{(l)}(\hat{\beta}) \left( \rho_{(ij)t-1,l} + (\alpha + \beta)^{t-T_{l-1}-2} c_{t-1,l} \right) \\
&- E_{(l)} \left[ \hat{\beta} \left( \frac{1}{T} \sum_{m=1}^{k+1} (T_m - T_{m-1}) \rho_{(ij)t-1,m} + \frac{1}{T} \sum_{l=1}^{k+1} \sum_{t=T_{l-1}+1}^{T_l} (\alpha + \beta)^{t-T_{l-1}-2} c_{t-1,l} \right. \right. \\
&\left. \left. + \frac{1}{T} \sum_{l=1}^{k+1} \sum_{t=T_{l-1}+1}^{T_l} r_{t-1} \right) \right] + o(1)_T
\end{aligned}$$

where  $c_l$  represents the distance of the initial value of the segment  $l$  from the unconditional mean for that segment as in Lemma 1

$$c_{t,l} = \sum_{s=1}^{\infty} \beta^{s-1} \left[ (1 - \alpha - \beta) \rho_{ij} + \alpha z_{i,T_{l-1}-s+1}^* z_{j,T_{l-1}-s+1}^* \right]$$

and  $r_{ij,t}$  represents the  $(i, j)$ th element of  $R_t$  in (13). By a law of large numbers,

$$\frac{1}{T_l - T_{l-1}} \sum_{t=T_{l-1}+1}^{T_l} r_t = o(1)_{T_l - T_{l-1}}.$$

Moreover, given that  $\text{cov}(\hat{\alpha}, z_{i,t}^* z_{j,t}^*) = o(1)_T$  and  $\text{cov}(\hat{\beta}, z_{i,t}^* z_{j,t}^*) = o(1)_T$ , the above expression becomes

$$\begin{aligned}
\rho_{(ij)t,l} - \frac{1}{T} \sum_{m=1}^{k+1} (T_m - T_{m-1}) \rho_{(ij)t,m} &= E_{(l)}(\hat{\alpha} + \hat{\beta}) \left[ \rho_{(ij)t-1,l} - \frac{1}{T} \sum_{m=1}^{k+1} (T_m - T_{m-1}) \rho_{(ij)t-1,m} \right] \\
&+ o(1)_T
\end{aligned}$$

Since  $\rho_{(ij)t,l} - \frac{1}{T} \sum_{m=1}^{k+1} (T_m - T_{m-1}) \rho_{(ij)t,m} \neq 0$  in the presence of at least a structural break,  $E_{(l)}(\hat{\alpha} + \hat{\beta}) = 1$  as  $T_l - T_{l-1} \rightarrow \infty$ . ■